

Fibers of Polynomial Mappings Over \mathbb{R}^n

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Abstract

We prove results on fibers of polynomial mappings $\mathbb{R}^n \rightarrow \mathbb{R}^n$ and deduce when such mappings are surjective under certain conditions.

1 The results

Definition 1.1. Let $p(X_1, \dots, X_n) \in \mathbb{R}[X_1, \dots, X_n]$. We denote by $\bar{p}(X_1, \dots, X_n)$ the leading homogeneous component of $p(X_1, \dots, X_n)$ with respect to the standard grading, $\deg X_j = 1$ for $1 \leq j \leq n$.

Theorem 1.2. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $f(X_1, \dots, X_n) = (p_1(X_1, \dots, X_n), \dots, p_n(X_1, \dots, X_n))$ be a polynomial mapping (i.e. $(p_1, \dots, p_n) \in \mathbb{R}[X_1, \dots, X_n]^n$). Let $g_{ij}(X_1, \dots, X_n) \in \mathbb{R}[X_1, \dots, X_n]$, $i, j = 1, \dots, n$. Let $(\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}^+)^n$. We assume that the following 2 conditions hold true:

- (i) The determinant $\det(g_{ij}(X_1, \dots, X_n))_{i,j=1,\dots,n}$ never vanishes in \mathbb{R}^n .
- (ii) The following system of n equations in n unknowns is such that the degree of each of the equations is an odd number:

$$\sum_{j=1}^n (p_j(X_1, \dots, X_n))^{\alpha_j} g_{ij}(X_1, \dots, X_n) = 0, \quad i = 1, \dots, n. \quad (1.1)$$

Then the following 2 assertions are true:

- (a) If the induced homogeneous system of the system (1.1):

$$\sum_{j=1}^n (\bar{p}_j(X_1, \dots, X_n))^{\alpha_j} \bar{g}_{ij}(X_1, \dots, X_n) = 0, \quad i = 1, \dots, n, \quad (1.2)$$

has only the zero solution $(X_1, \dots, X_n) = (0, \dots, 0)$ over \mathbb{R} , then $f(\mathbb{R}^n) = \mathbb{R}^n$.

(b) If the induced homogeneous system of the system in equation (1.1), i.e. the system (1.2) has only the zero solution over \mathbb{C} , then $\forall (a_1, \dots, a_n) \in \mathbb{R}^n$, either $|f^{-1}(a_1, \dots, a_n)| = \infty$ over \mathbb{C} under the extra assumption that $\det(g_{ij}(Z_1, \dots, Z_n))_{i,j=1,\dots,n} \in \mathbb{R}^\times$, or there exists an integer $k = k(a_1, \dots, a_n) \geq 0$ such that $|f^{-1}(a_1, \dots, a_n)| = 2k + 1$ over \mathbb{R} .

Proof.

(a) Let $(a_1, \dots, a_n) \in \mathbb{R}^n$. We will prove that $(a_1, \dots, a_n) \in f(\mathbb{R}^n)$. we consider the following system of equations:

$$X_{n+1}^{d_i} \sum_{j=1}^n \left(p_j \left(\frac{X_1}{X_{n+1}}, \dots, \frac{X_n}{X_{n+1}} \right) - a_j \right)^{\alpha_j} g_{ij} \left(\frac{X_1}{X_{n+1}}, \dots, \frac{X_n}{X_{n+1}} \right) = 0, \quad (1.3)$$

$$\text{where } d_i = \deg \left(\sum_{j=1}^n (p_j)^{\alpha_j} g_{ij} \right), \quad i = 1, \dots, n.$$

This is a system of n homogeneous real polynomial equations in the $n + 1$ unknowns X_1, \dots, X_n, X_{n+1} , and by condition (ii) the degrees d_i , $i = 1, \dots, n$ of all of these equations are odd integers. By well known facts on varieties over \mathbb{R} (see [1]), it follows that the system (1.3) has a non-zero real solution $(X_1, \dots, X_n, X_{n+1}) = (X_1^0, \dots, X_n^0, X_{n+1}^0)$. We must have $X_{n+1}^0 \neq 0$, for otherwise $(X_1^0, \dots, X_n^0) \neq (0, \dots, 0)$ and (X_1^0, \dots, X_n^0) is a solution of (1.1), i.e. (1.2). This contradicts the assumption of the theorem in part (a). Thus we get from equation (1.3):

$$\sum_{j=1}^n \left(p_j \left(\frac{X_1^0}{X_{n+1}^0}, \dots, \frac{X_n^0}{X_{n+1}^0} \right) - a_j \right)^{\alpha_j} g_{ij} \left(\frac{X_1^0}{X_{n+1}^0}, \dots, \frac{X_n^0}{X_{n+1}^0} \right) = 0, \quad \text{for } i = 1, \dots, n. \quad (1.4)$$

By condition (i) of our theorem, this implies that

$$f \left(\frac{X_1^0}{X_{n+1}^0}, \dots, \frac{X_n^0}{X_{n+1}^0} \right) = (a_1, \dots, a_n).$$

(b) Let us consider the system (1.3) over \mathbb{C} . By the Bezout Theorem (see [1]), either the system (1.3) has infinitely many solutions over \mathbb{C} , or it has exactly

$$\prod_{i=1}^n \deg \left(\sum_{j=1}^n (p_j)^{\alpha_j} g_{ij} \right)$$

solutions over \mathbb{C} , counting multiplicities and not counting the zero solution. In the case we have infinitely many solutions over \mathbb{C} , we must have for each such a solution $(Z_1^0, \dots, Z_n^0, Z_{n+1}^0)$ that $Z_{n+1}^0 \neq 0$, for by the assumption in part (b) of our theorem, the induced homogeneous system (1.2), of the system (1.1) has only the zero solution over \mathbb{C} . Since we also assume in this case that $\det(g_{ij}(Z_1, \dots, Z_n))_{i,j=1,\dots,n} \in \mathbb{C}^\times$ it follows as before by equation (1.4) that the fiber over \mathbb{C} , $f^{-1}(a_1, \dots, a_n)$ contains infinitely many points:

$$\left(\frac{Z_1^0}{Z_{n+1}^0}, \dots, \frac{Z_n^0}{Z_{n+1}^0} \right).$$

In the second case, in which we have exactly

$$\prod_{i=1}^n \deg \left(\sum_{j=1}^n (p_j)^{\alpha_j} g_{ij} \right)$$

solutions over \mathbb{C} , noting that by condition (ii) this number is an odd integer and that non-real solutions $(Z_1^0, \dots, Z_n^0, Z_{n+1}^0)$ come in conjugate pairs, we deduce that the fiber over \mathbb{R} , $f^{-1}(a_1, \dots, a_n)$, contains an odd number of points. \square

Corollary 1.3. *Let the polynomial mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, be given by $f(X_1, \dots, X_n) = (p_1(X_1, \dots, X_n), \dots, p_n(X_1, \dots, X_n))$. Let $g_{ij}(X_1, \dots, X_n) \in \mathbb{R}[X_1, \dots, X_n]$ for $i, j = 1, \dots, n$. Let $(\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}^+)^n$. We assume that the following two conditions hold true:*

- (i) *The determinant $\det(g_{ij}(X_1, \dots, X_n))_{i,j=1,\dots,n}$ never vanishes in \mathbb{R}^n .*
- (ii) *For each $i = 1, \dots, n$ the set $\{\alpha_j \deg p_j + \deg g_{ij} \mid j = 1, \dots, n\}$ contains a unique maximal element $\alpha_{j(i)} \deg p_{j(i)} + \deg g_{ij(i)}$, which is an odd integer. We agree that $\deg 0 = -\infty$.*

Let us consider the following homogeneous system:

$$\bar{p}_{j(i)} \bar{g}_{ij(i)} = 0, \quad i = 1, \dots, n. \quad (1.5)$$

Then the following two assertions are true:

- (a) *If the system (1.5) has only the zero solution over \mathbb{R} , then $f(\mathbb{R}^n) = \mathbb{R}^n$.*
- (b) *If the system (1.5) has only the zero solution over \mathbb{C} , then for any $(a_1, \dots, a_n) \in \mathbb{R}^n$ either $|f^{-1}(a_1, \dots, a_n)| = \infty$ over \mathbb{C} , provided that also the following assumption holds true, $\det g_{ij}(Z_1, \dots, Z_n)_{i,j=1,\dots,n} \in \mathbb{R}^\times$, or that there exists an integer $k = k(a_1, \dots, a_n) \geq 0$ such that $|f^{-1}(a_1, \dots, a_n)| = 2k + 1$ over \mathbb{R} .*

Proof.

This is a special case of Theorem 1.2, where the system (1.5) is precisely the system (1.2) because of the maximality and the uniqueness of $\alpha_{j(i)} \deg p_{j(i)} + \deg g_{ij(i)}$ among the elements of the set $\{\alpha_j \deg p_j + \deg g_{ij} \mid j = 1, \dots, n\}$. \square

Corollary 1.4. *Let the polynomial mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, be given by $f(X_1, \dots, X_n) = (p_1(X_1, \dots, X_n), \dots, p_n(X_1, \dots, X_n))$. Suppose that the product $(\deg p_1) \cdot \dots \cdot (\deg p_n)$ is an odd integer. Then the following two assertions are true:*

- (a) *If $|\bar{f}^{-1}(0, \dots, 0)| = 1$ over \mathbb{R} , then $f(\mathbb{R}^n) = \mathbb{R}^n$.*
- (b) *If $|\bar{f}^{-1}(0, \dots, 0)| = 1$ over \mathbb{C} , then $\forall (a_1, \dots, a_n) \in \mathbb{R}^n$ either the fiber size $|f^{-1}(a_1, \dots, a_n)| = \infty$ over \mathbb{C} , or there exists an integer $k = k(a_1, \dots, a_n) \geq 0$ such that $|f^{-1}(a_1, \dots, a_n)| = 2k + 1$ over \mathbb{R} .*

Proof.

This follows by Corollary 1.3, where $(\alpha_1, \dots, \alpha_n) = (1, \dots, 1)$ and where $g_{ij} = \delta_{ij}$, $i, j = 1, \dots, n$ because the system (1.5) becomes $\bar{p}_j = 0$, $j = 1, \dots, n$ which has the solution set $\bar{f}^{-1}(0, \dots, 0)$. \square

Remark 1.5. We note that if in Corollary 1.4 we have $\deg p_j = 1$, $j = 1, \dots, n$, i.e. if all the $p_j = \bar{p}_j$ are linear forms then we get the well known fact from linear algebra. Namely, if $A\bar{X} = \bar{0}$ is an $n \times n$ linear homogeneous system that has only the trivial solution, then $A\bar{X} = \bar{b}$ is consistent $\forall \bar{b} \in \mathbb{R}^n$.

Remark 1.6. If for $j = 1, \dots, n$, $b_j \geq 0$ is an integer and if we have

$$p_j(X_1, \dots, X_n) = \sum_{i=1}^n a_{ij} X_i^{2b_j+1} + \text{elements of degrees} < 2b_j + 1.$$

Then the polynomial mapping $f(X_1, \dots, X_n) = (p_1(X_1, \dots, X_n), \dots, p_n(X_1, \dots, X_n))$ is a surjective mapping, i.e. $f(\mathbb{R}^n) = \mathbb{R}^n$, provided that the only solution of the following system:

$$\sum_{i=1}^n a_{ij} X_i^{2b_j+1} = 0, \quad j = 1, \dots, n,$$

is the trivial solution: $X_1 = \dots = X_n = 0$.

For in this case the above system is the system (1.5) of Corollary 1.4 ($g_{ij} =$

δ_{ij}). For example, this is the case for the equal-degree case $b_1 = \dots = b_n = b$ provided that $\det(a_{ij})_{i,j=1,\dots,n} \neq 0$. Another example is the following: we pick 4 non-zero real numbers, a , b , c and d such that $\text{sgn}(ad) = -\text{sgn}(bc)$. Then any mapping of the form:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad f(X, Y) = (aX^{2k+1} + bY^{2k+1} + \dots, cX^{2j+1} + dY^{2j+1} + \dots),$$

is a surjective mapping. For the system (1.5) is:

$$\begin{cases} aX^{2k+1} + bY^{2k+1} &= 0 \\ cX^{2j+1} + dY^{2j+1} &= 0 \end{cases}.$$

If $k \leq j$ then the system can be written as follows:

$$\begin{cases} aX^{2k+1} + bY^{2k+1} &= 0 \\ (cX^{2(j-k)} + dY^{2(j-k)})Y^{2k+1} &= 0 \end{cases}.$$

We view this as a linear homogeneous system in the unknowns X^{2k+1} and Y^{2k+1} . Then the coefficients matrix is:

$$\begin{pmatrix} a & b \\ cX^{2(j-k)} & dY^{2(j-k)} \end{pmatrix}.$$

The determinant of this matrix is $(ad)Y^{2(j-k)} - (bc)X^{2(j-k)}$ and this can not be 0 because of the assumption $\text{sgn}(ad) = -\text{sgn}(bc)$, unless $j > k$ and $X = Y = 0$. In the other cases the only solution is, again, $X = Y = 0$.

Theorem 1.7. *Let $g_{ij}(X_1, \dots, X_n) \in \mathbb{R}[X_1, \dots, X_n]$ for $i, j = 1, \dots, n$ satisfy the condition that $\det(g_{ij}(X_1, \dots, X_n))_{i,j=1,\dots,n}$ never vanishes in \mathbb{R}^n . Then for any j_0 , $1 \leq j_0 \leq n$, such that the degrees $\deg g_{ij_0}$, $i = 1, \dots, n$ are all odd integers the system:*

$$\bar{g}_{ij_0}(X_1, \dots, X_n) = 0, \quad i = 1, \dots, n, \quad (1.6)$$

has non-zero real solutions.

Proof.

Let j_0 be such that the degrees $\deg g_{ij_0}$, $i = 1, \dots, n$, are all odd integers. In Corollary 1.3 we take the following:

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad f(X_1, \dots, X_n) = (\delta_{1j_0}, \dots, \delta_{j_0j_0}, \dots, \delta_{nj_0}).$$

$$\text{and } (\alpha_1, \dots, \alpha_n) = (1, \dots, 1).$$

Then conditions (i) and (ii) of Corollary 1.3, with the choice $j(i) = j_0$ are satisfied. Since $f(\mathbb{R}^n) \neq \mathbb{R}^n$ it must be that the system (1.5) has non-zero solutions over \mathbb{R} . But in this case the system (1.5) coincides with the system above, (1.6). \square

Theorem 1.8. *Let the polynomial mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, be given by $f(X_1, \dots, X_n) = (p_1(X_1, \dots, X_n), \dots, p_n(X_1, \dots, X_n))$. Suppose that the determinant $\det J(f)(X_1, \dots, X_n)$ never vanishes in \mathbb{R}^n . Then $\forall j, 1 \leq j \leq n$ the system:*

$$\overline{p_j \frac{\partial p_j}{\partial X_i}} = 0, \quad i = 1, \dots, n, \quad (1.7)$$

has non-trivial solutions over \mathbb{R} .

Proof.

Let $j = j_0$ be such that the system (1.7) has only the zero solution over \mathbb{R} . We will arrive at a contradiction by showing that this assumption implies on the one hand, $f(\mathbb{R}^n) = \mathbb{R}^n$, and it also implies, on the other hand, $f(\mathbb{R}^n) \neq \mathbb{R}^n$.

1) We first prove that $f(\mathbb{R}^n) = \mathbb{R}^n$. To see that, we use Corollary 1.3 with:

$$g_{ij}(X_1, \dots, X_n) = \frac{\partial p_j}{\partial X_i} \quad \text{for } i, j = 1, \dots, n.$$

We can assume without losing the generality that:

$$\deg g_{ij_0} = \deg p_{j_0} - 1, \quad i = 1, \dots, n. \quad (1.8)$$

For the assumptions of our theorem as well as the conclusion $f(\mathbb{R}^n) = \mathbb{R}^n$, are invariant with respect to a real, non-singular change of the variables. More precisely, instead of working with the original mapping, $f(X_1, \dots, X_n)$, we could have, first performed a change of the variables, as follows:

$$X_j = \sum_{i=1}^n a_{ij} U_i, \quad j = 1, \dots, n, \quad (1.9)$$

where $(a_{ij})_{i,j=1,\dots,n}$ is a real non-singular matrix. Then we could have proved that the mapping given by $F(U_1, \dots, U_n) = f(X_1, \dots, X_n)$ is epimorphic and that would have implied that the original mapping $f(X_1, \dots, X_n)$ is epimorphic. The linear transformation we choose in equation (1.9) is such that $a_{ij} \neq 0$ for all $i, j = 1, \dots, n$. With this choice of the linear transformation it is clear that generically (in the $a_{ij} \neq 0$), each of the components $\tilde{p}_j(U_1, \dots, U_n) = p_j(X_1, \dots, X_n)$, $j = 1, \dots, n$, of the mapping $F(U_1, \dots, U_n)$ has the property that for each $i = 1, \dots, n$ it contains all the monomials of the form $a U_1^{m_1} \dots U_n^{m_n}$ where $a \neq 0$, and where $\sum_{k=1}^n m_k = \deg p_j$, and $m_i \neq 0$. This justifies equation (1.8). Next we choose in Corollary 1.3 the following: For $j \neq j_0$ we take $\alpha_j = 1$. We choose

the positive integer α_{j_0} so large that $\alpha_{j_0} \deg p_{j_0} + (\deg p_{j_0} - 1)$ is strictly larger than $\deg p_j + \deg g_{ij}$ for $i = 1, \dots, n$ and $j \neq j_0$. Also α_{j_0} is such that $\alpha_{j_0} \deg p_{j_0} + (\deg p_{j_0} - 1)$ is an odd integer. That is always possible to do: If $\deg p_{j_0}$ is an even integer, then there is no other restriction on α_{j_0} (except for being large enough). If $\deg p_{j_0}$ is an odd integer, then α_{j_0} must also be an odd integer. Now conditions (i) and (ii) of Corollary 1.3 are satisfied with $j(i) = j_0$. The system (1.5) Corollary 1.3 reduces to the system (1.7) with $j = j_0$ and so by part (a) of Corollary 1.3 it follows that $f(\mathbb{R}^n) = \mathbb{R}^n$.

2) In order to conclude the proof of Theorem 1.8, we now prove that the existence of such a j_0 implies that $f(\mathbb{R}^n) \neq \mathbb{R}^n$. We may assume that $\bar{p}_{j_0}(X_1, \dots, X_n) \geq 0 \ \forall (X_1, \dots, X_n) \in \mathbb{R}^n$, and there is an equality $\bar{p}_{j_0}(X_1^0, \dots, X_n^0) = 0$ if and only if $(X_1^0, \dots, X_n^0) = (0, \dots, 0)$. Let us denote $d = \deg \bar{p}_{j_0}$. We claim that $\forall i, 1 \leq i \leq n$ we have $\deg_{X_i} \bar{p}_{j_0} = d$:

For let $\bar{p}_{j_0}(X_1, \dots, X_n) = \sum_{k=0}^N h_k(X_1, \dots, \hat{X}_i, \dots, X_n) X_i^k$ where h_k is an homogeneous polynomial in $(X_1, \dots, \hat{X}_i, \dots, X_n)$ of degree $d - k$. Then $\bar{p}_{j_0}(0, \dots, 0, X_i, 0, \dots, 0) \equiv 0$ for any choice of X_i which is impossible. Hence we obtain:

$$\bar{p}_{j_0}(X_1, \dots, X_n) = \sum_{i=1}^n \lambda_i X_i^d + h(X_1, \dots, X_n), \quad (1.10)$$

where $\lambda_i > 0, \forall i, 1 \leq i \leq n$ and where h is homogeneous of degree d such that $\deg_{X_i} h < d, \forall i, 1 \leq i \leq n$. Since $\bar{p}_{j_0} \geq 0$ it follows that d is an even integer and now equation (1.10) implies the existence of an $M > 0$ such that $\forall (X_1, \dots, X_n) \in \mathbb{R}^n$ we have $p_{j_0}(X_1, \dots, X_n) \geq -M$. Hence we conclude that $f(\mathbb{R}^n) \neq \mathbb{R}^n$. Now the proof of the theorem is completed. \square

Theorem 1.9. *Let the polynomial mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, be given by $f(X_1, \dots, X_n) = (p_1(X_1, \dots, X_n), \dots, p_n(X_1, \dots, X_n))$. Suppose that the determinant $\det J(f)(X_1, \dots, X_n)$ never vanishes in \mathbb{R}^n . If there is an even integral vector $(\alpha_1, \dots, \alpha_n) \in (2\mathbb{Z}^+)^n$ such that the induced homogeneous system of:*

$$\sum_{j=1}^n \alpha_j \cdot (p_j(X_1, \dots, X_n))^{\alpha_j-1} \frac{\partial p_j}{\partial X_i} = 0, \quad i = 1, \dots, n, \quad (1.11)$$

has only the zero solution over \mathbb{R} , then $f(\mathbb{R}^n) = \mathbb{R}^n$.

Proof.

Let us consider the following polynomial: $F(X_1, \dots, X_n) = \sum_{j=1}^n (p_j(X_1, \dots, X_n))^{\alpha_j}$. Since $p_j(X_1, \dots, X_n) \in \mathbb{R}[X_1, \dots, X_n], \forall j = 1, \dots, n$ and since the vector

$(\alpha_1, \dots, \alpha_n)$ is an even integral vector, it follows that $\deg F$ is an even integer. Say $\deg F = 2N$ for some $N \in \mathbb{Z}^+$. Clearly, the assumptions as well as the conclusion of Theorem 1.9 are invariant with respect to a real non-singular linear change of the variables. Thus, as we explained in the proof of Theorem 1.8 we can assume that:

$$\deg \left(\frac{\partial F}{\partial X_i} \right) = \deg F - 1 = 2N - 1, \quad i = 1, \dots, n.$$

Let us take in Theorem 1.2:

$$g_{ij}(X_1, \dots, X_n) = \frac{\partial p_j}{\partial X_i}, \quad \text{for } i, j = 1, \dots, n.$$

The vector of integers in Theorem 1.2 will be $(\alpha_1 - 1, \dots, \alpha_n - 1)$, and the j 'th component of the mapping in Theorem 1.2 will be:

$$(\alpha_j)^{1-\alpha_j^{-1}} p_j(X_1, \dots, X_n).$$

These satisfy the conditions (i) and (ii) of Theorem 1.2 and now part (a) of Theorem 1.2 implies that $f(\mathbb{R}^n) = \mathbb{R}^n$. \square

Pinchuk's example.

Pinchuk defined the following:

$$t = xy - 1, \quad s = 1 + xt, \quad h = ts, \quad f = s^2(t^2 + y),$$

and then set,

$$p = h + f, \quad q = -t^2 - 6th(h + 1) - u(f, h),$$

where

$$u = A(h)f + B(h),$$

$$A = h + \frac{1}{45}(13 + 15h)^3,$$

$$B = 4h^3 + 6h^2 + \frac{1}{2}h^2 + \frac{1}{2700}(13 + 15h)^4.$$

Thus we have:

$$\deg h = 5, \quad \deg f = 10, \quad \deg p = 10, \quad \deg q = 25.$$

Pinchuk's example is the following mapping:

$$(p, q) = (x^6 y^4 - 2x^5 y^3 + \dots, \frac{15^3}{45} x^{15} y^{10} + \dots).$$

We are interested only in the leading homogeneous components. Thus:

$$p = x^6 y^4 + \dots, \quad \frac{\partial p}{\partial x} = 6x^5 y^4 + \dots, \quad \frac{\partial p}{\partial y} = 4x^6 y^3 + \dots$$

$$q = \frac{15^3}{45} x^{15} y^{10} + \dots, \quad \frac{\partial q}{\partial x} = \frac{15^4}{45} x^{14} y^{10} + \dots, \quad \frac{15^3 \cdot 10}{45} x^{15} y^9 + \dots$$

There are, in this case, two homogeneous systems of equations in (1.7) of Theorem 1.8:

$$\bar{p} \frac{\partial \bar{p}}{\partial x} = \bar{p} \frac{\partial \bar{p}}{\partial y} = 0,$$

and

$$\bar{q} \frac{\partial \bar{q}}{\partial x} = \bar{q} \frac{\partial \bar{q}}{\partial y} = 0.$$

These reduce to:

$$\begin{array}{rcl} x^{11} y^8 & = & x^{12} y^7 = 0 \\ x^{29} y^{20} & = & x^{30} y^{19} = 0 \end{array}.$$

Thus both systems have non-zero solutions:

$$\{(0, y) \mid y \in \mathbb{R}\} = \{(x, 0) \mid x \in \mathbb{R}\},$$

as should be the case according to Theorem 1.8.

Remark 1.10. The Pinchuk construction gives coordinates with a single element as their highest homogeneous component. This element has the form $\alpha x^m y^k$ where $\alpha \in \mathbb{R}^\times$, $m, k \geq 1$. Thus the equations in (1.7) of Theorem 1.8 are of the form:

$$x^m y^k \cdot x^{m-1} y^k = x^m y^k \cdot x^m y^{k-1} = 0,$$

i.e.

$$x^{2m-1} y^{2k} = x^{2m} y^{2k-1} = 0,$$

and so the solution set is the union of both axis:

$$\{(0, y) \mid y \in \mathbb{R}\} = \{(x, 0) \mid x \in \mathbb{R}\},$$

which, of course, is non-trivial in agreement with Theorem 1.8.

References

- [1] I. R. Shafarevich, Basic Algebraic Geometry, Verlag-Springer, 1970.

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